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The Peetre K-Functional and the Riesz Summability Operator for the Fourier–Legendre Expansions

Luoqing Li*

Department of Mathematics, Hubei University, Wuhan 430062, People's Republic of China
E-mail: lilq@hubu.edu.en

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The Peetre K-functionals and the generalized Riesz summability operators are introduced. The convergence and boundedness of the Riesz operators are discussed. The equivalent relationships of the Peetre K-functionals and the Riesz operators are established. © 1999 Academic Press

1. INTRODUCTION AND NOTATIONS

Let $L_p[-1, 1]$, $1 \leqslant p < \infty$, denote the spaces of the Lebesgue integrable functions on $[-1, 1]$, and let $C[-1, 1]$ denote the space of the continuous functions on $[-1, 1]$, with the norms

$$\|f\|_p := \left\{\int_{-1}^{1} |f(x)|^p \, dx\right\}^{1/p}, \qquad \text{for} \quad f \in L_p[-1, 1], \qquad \text{and}$$

$$\|f\|_\infty := \sup_{-1 \leqslant x \leqslant 1} |f(x)|, \qquad \text{for} \quad f \in C[-1, 1],$$

respectively. In the following, $L_p[-1, 1]$ will always be one of the spaces $L_p[-1, 1]$ for $1 \leqslant p < \infty$, or $C[-1, 1]$ for $p = \infty$. Let Π_n be the class of polynomials of degree $\leqslant n$. The best polynomial approximant of degree n of $f \in L_p[-1, 1]$ is defined by

$$E_n(f)_p := \inf\{\|f - p_n\|_p : p_n \in \Pi_n\}.$$

Z. Ditzian and V. Totik [4, Chap. 7] constructed a polynomial $p_n \in \Pi_n$ satisfying

$$\|f - p_n\|_p \leqslant K_{r,\varphi}(f, n^{-r})_p, \tag{1.1}$$

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where the Peetre K-functional $K_{r,\varphi}(f, n^{-r})_p$ with weight $\varphi(x) = \sqrt{1-x^2}$ is defined by

$$K_{r,\varphi}(f, t^r)_p := \inf\{\|f-g\|_p + t^r \|\varphi^r g^{(r)}\|_p : g \in C^r[-1, 1]\}. \tag{1.2}$$

Result (1.1) implies that

$$E_n(f)_p \leqslant K_{r,\varphi}(f, n^{-r})_p.$$

The Peetre K-functional is a very useful tool for estimating the rate of convergence of linear operators. Recently, Z. Ditzian and K. Ivanov [3] and V. Totik [6, 7], etc., considered some strong converse inequalities of approximation by linear operators. Their results show that the order for approximation by some linear operators is completely characterized by the corresponding K-functional, which is equivalent to the some moduli of smoothness. For example, for Bernstein operators

$$B_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \qquad b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

V. Totik [7] has proved that

$$\|f - B_n(f)\|_{C[0,1]} \simeq K_{2,\varphi}(f, n^{-1})_\infty, \tag{1.3}$$

where the weight function $\varphi(x) = \sqrt{x(1-x)}$, and $A \simeq B$ means there exists a positive constant "const" such that (1/const) $A \leqslant B \leqslant \text{const}\, A$. In this paper, we denote "const" an absolute positive constant which is dependent only on the parameters indicated by the index.

For Bernstein–Durrmeyer operators

$$M_n(f, x) := \frac{1}{n+1} \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 b_{n,k}(y) f(y)\, dy,$$

W. Chen *et al.* [1] proved that

$$\|(M_n - I)^r f\|_{L_p[0,1]} \simeq K_{2r}(f, n^{-r})_p, \tag{1.4}$$

where I is the identity and the Peetre K-functional is defined by

$$K_{2r}(f, n^{-2r})_p := \inf\{\|f-g\|_{L_p[0,1]} + n^{-2r} \|P_1(D)^r g\|_{L_p[0,1]} : g \in C^{2r}[0, 1]\},$$

and the differential operator $P_1(D) := (d/dx)\, x(1-x)(d/dx)$.

In his paper, Z. Ditzian [2] considered the Riesz summability operators R_n for Fourier–Legendre expansions. Let $P_k(x)$ be the Legendre polynomials, and let differential operator $P(D) := (d/dx)\ (1-x^2)(d/dx)$. The formal Fourier–Legendre expansion of $f \in L_1[-1, 1]$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} f^\wedge(n)\, P_n(x), \tag{1.5}$$

where $f^\wedge(n)$ is the Legendre transform of f defined by

$$f^\wedge(n) := \langle f, P_n \rangle = \int_{-1}^{1} f(y)\, P_n(y)\, dy, \qquad n = 0, 1, \ldots .$$

For series (1.5) Z. Ditzian [2] defined the Riesz operators

$$R_n(f, x) := \sum_{k=0}^{n} \left(1 - \frac{k(k+1)}{n(n+1)}\right) f^\wedge(k)\, P_k(x),$$

and proved that

$$\|(R_n - I)^r f\|_p \simeq K_{2r}(f, n^{-2r})_p, \tag{1.6}$$

where I is the identity and the Peetre K-functional is given by

$$K_{2r}(f, t^{2r})_p := \inf\{\|f - g\|_p + t^{2r}\, \|P(D)^r g\|_p : g \in C^{2r}[-1, 1]\}. \tag{1.7}$$

Combining (1.4) with (1.7) yields the inequality

$$E_n(f)_p \leqslant \text{const}_p\, \|(M_{n^2} - I)^r f\|_p, \qquad 1 \leqslant p \leqslant \infty,\ r \geqslant 1.$$

The aim of the paper is to consider the generalized Riesz summability operators

$$R_{n,r}(f, x) := \sum_{k=0}^{n} \left(1 - \left(\frac{k(k+1)}{n(n+1)}\right)^{r/2}\right) f^\wedge(k)\, P_k(x), \qquad r \geqslant 1, \quad n = 0, 1, \ldots \tag{1.8}$$

We will establish the equivalence

$$\|R_{n,r} f - f\|_p \simeq K(f, n^{-r};\, L_p, W_p^r). \tag{1.9}$$

The definition of the Peetre K-functional $K(f, n^{-r}; L_p, W_p^r)$ will be given in Section 2.

Remark. The equivalence result (1.9) is different from that of (1.6) in two respects. We deal with $R_{n,r} - I$ instead of the power of the operators $(R_n - I)^r$ and the r in our definition is not restricted to be natural numbers. The proof follows closely that of [2], presented by Z. Ditzian.

2. A PEETRE K-FUNCTIONAL FOR FOURIER–LEGENDRE EXPANSIONS

Let $P(D) = (d/dx)(1 - x^2)(d/dx)$. The Legendre polynomials $P_k(x)$ are given by

$$P(D)\, P_k(x) = -k(k+1)\, P_k(x),$$

and satisfy the orthonormality condition

$$\langle P_k, P_m \rangle = \int_{-1}^{1} P_k(x)\, P_m(x)\, dx = \delta_{k,m}.$$

See [5] for more details.

We deduce the fractional derivative for the expansion (1.5). Let $\mathscr{D}^r := -(-P(D))^{r/2}$ be the power of the operator $P(D)$ given by the relation

$$\mathscr{D}^r P_k(x) = -(k(k+1))^{r/2}\, P_k(x). \tag{2.1}$$

Let

$$W_p^r := \{ f \in L_p[-1,1] : \exists g \in L_p[-1,1] \ni \forall k \in \mathbf{N}_0,\ g^\wedge(k) = -(k(k+1))^{r/2}\, f^\wedge(k) \}.$$

Therefore if $f \in W_p^r$ has the formal Fourier–Legendre expansion (1.5)

$$f(x) \sim \sum_{k=0}^{\infty} f^\wedge(k)\, P_k(x),$$

then $\mathscr{D}^r f \in L_p[-1,1]$ and has the following formal Fourier–Legendre expansion

$$\mathscr{D}^r f(x) \sim \sum_{k=0}^{\infty} [-(k(k+1))^{r/2}]\, f^\wedge(k)\, P_k(x).$$

The Peetre K-functional between $L_p[-1,1]$ and W_p^r is then defined by

$$K(f, t^r; L_p, W_p^r) := \inf\{ \|f - g\|_p + t^r \|\mathscr{D}^r g\|_p,\ g \in W_p^r \}. \tag{2.2}$$

Since W_p^r is dense in $L_p[-1,1]$ we have $K(f, t^r; L_p, W_p^r) \to 0$ as $t \to 0$. It is suitable to measure the rate of convergence of the generalized Riesz summability operators by the Peetre K-functional $K(f, t^r; L_p, W_p^r)$.

3. EQUIVALENCE RESULTS

The following equivalence relation is a strong converse inequality in the sense of [3].

THEOREM 1. *Let* $f \in L_p[-1, 1], 1 \leqslant p \leqslant \infty$, *and let* $R_{n,r}(f, x)$ and $K(f, n^{-r}; L_p, W_p^r)$ *be defined by* (1.8) *and* (2.2), *respectively. We have the equivalence relation*

$$\| R_{n,r} f - f \|_p \simeq K(f, n^{-r}; L_p, W_p^r). \tag{3.1}$$

In order to prove the theorem we give the following three lemmas.

LEMMA 1. *Let* $1 \leqslant p \leqslant \infty$, *and let* $R_{n,r}$ *be defined by* (1.8). *Then* $R_{n,r}$ *is of type* (p, p), *i.e.*,

$$\| R_{n,r} f \|_p \leqslant \text{const}_{p,r} \| f \|_p, \qquad f \in L_p[-1, 1]. \tag{3.2}$$

Proof. If $r = 2$, Z. Ditzian [2] gave a proof for (3.2). If $r \neq 2$, we can deduce (3.2) from theorem 3.9 in [8] by using multiplier theory. Let

$$J_k f(x) = f^\wedge(k)\, P_k(x).$$

Then $\{J_k\}_{k=0}^{\infty}$ is a total, fundamental system of mutually orthogonal projections satisfying

$$\| (C, 1)_n f \|_p \leqslant \text{const}_p \| f \|_p, \qquad f \in L_p[-1, 1],$$

where $(C, 1)_n (f, x)$ is the Cesàro means

$$(C, 1)_n (f, x) := \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) f^\wedge(k)\, P_k(x).$$

In order to verify (3.2), we use Theorem 3.9 of [8] and choose $j = 1$, $\Phi(t) = \Psi(t) = t(t+1)$ and $e(x) = 1 - x^{r/2}$ for $0 \leqslant x \leqslant 1$ or $e(x) = 0$ for $x > 1$. We have to show that $e(x)$ satisfies $\int_0^\infty x^2 |de''(x)| < \infty$. This is easy to check. In fact, we have

$$\int_0^\infty x^2 |de''(x)| = \frac{1}{4} r\, |r-2| \int_0^1 x^{r/2}\, dx = \frac{r\, |r-2|}{2(r+2)} < \infty.$$

Therefore all the conditions in Theorem 3.9 of [8] are satisfied. Then $\{1 - (k(k+1)/n(n+1))^{r/2}\}_{k=0}^{n}$ is a family of uniformly bounded multipliers on $L_p[-1, 1]$. This completes the proof of (3.2).

As a corollary of Lemma 1, we have $\lim_{n \to \infty} \| R_{n,r} f - f \|_p = 0$ for all $f \in L_p[-1, 1]$. That is to say $\{R_{n,r}\}$ is an approximation process on $L_p[-1, 1]$.

Similarly to [2], we obtain the following lemma which gives the relationships between the Riesz summability operators $R_{n,r}$ and the differential operator $\mathscr{D}^r$.

LEMMA 2. *Let* $f \in L_p[-1, 1]$, $1 \leqslant p \leqslant \infty$, *and let* $R_{n,r} f$ *be defined by* (1.8). *Then*

$$(n(n+1))^{r/2} R_{n,r}(R_{n,r} f - f) = \mathscr{D}^r R_{n,r} f. \tag{3.3}$$

Proof. We first note that for $f \in L_p[-1, 1]$ there holds

$$R_{n,r}(R_{n,r} f - f) = -\frac{1}{(n(n+1))^{r/2}} \times \sum_{k=0}^{n} \left(1 - \left(\frac{k(k+1)}{n(n+1)}\right)^{r/2}\right) (k(k+1))^{r/2} f^\wedge(k)\, P_k \tag{3.4}$$

for $0 \leqslant k \leqslant n$. By the definition of $\mathscr{D}^r$ we have

$$\mathscr{D}^r P_k(x) = -(k(k+1)^{r/2} P_k(x).$$

It follows that

$$\mathscr{D}^r R_{n,r}(f, x) = -\sum_{k=0}^{n} \left(1 - \left(\frac{k(k+1)}{n(n+1)}\right)^{r/2}\right) (k(k+1))^{r/2} f^\wedge(k)\, P_k(x).$$

Combining this equation with (3.4) we get

$$(n(n+1))^{r/2} R_{n,r}(R_{n,r} f - f) = \mathscr{D}^r R_{n,r} f.$$

Lemma 2 is proved.

For a given function in W_p^r we have the Jackson-type inequality by following an idea of Ditzian [2].

LEMMA 3. *Let* $f \in W_p^r$, $1 \leqslant p \leqslant \infty$, *and let* $R_{n,r} f$ *be defined by* (1.8). *Then*

$$\| R_{n,r} f - f \|_p \leqslant \frac{\text{const}_{p,r}}{n^r} \| \mathscr{D}^r f \|_p. \tag{3.5}$$

Proof. For $f \in L_p[-1, 1]$ we have from (3.3) in Lemma 2

$$R_{n,r}^2 f - R_{n,r} f = \frac{1}{(n(n+1))^{r/2}} \mathscr{D}^r R_{n,r} f.$$

By direct calculations, we know that $\mathscr{D}^r$ and $R_{n,r}$ commute, that is,

$$\mathscr{D}^r R_{n,r} f = R_{n,r} \mathscr{D}^r f, \qquad f \in W_p^r, \tag{3.6}$$

and for all n, m,

$$R_{m,r}\, R_{n,r} f = R_{n,r}\, R_{m,r} f, \qquad f \in L_p[-1, 1].$$

Furthermore

$$R_{m,r}^2 f - R_{m+1,r}\, R_{m,r} f = -\frac{(m+2)^{r/2} - m^{r/2}}{(m(m+1)(m+2))^{r/2}} \mathscr{D}^r R_{m,r} f,$$

and

$$R_{m+1,r}^2 f - R_{m,r}\, R_{m+1,r} f = \frac{(m+2)^{r/2} - m^{r/2}}{(m(m+1)(m+2))^{r/2}} \mathscr{D}^r R_{m+1,r} f.$$

Note that $(m+2)^{r/2} - m^{r/2} \simeq r m^{(r/2)-1}$ as $m \to \infty$. It follows that

$$\| R_{n,r}^2 f - R_{m+1,r}^2 f \|_p \leqslant \frac{\mathrm{const}_{p,r}}{m^{r+1}} (\| \mathscr{D}^r R_{m,r} f \|_p + \| \mathscr{D}^r R_{m+1,r} f \|_p).$$

Hence Lemma 1 and (3.6) yield for $f \in W_p^r$ that

$$\| R_{n,r}^2 f - R_{m+1,r}^2 f \|_p \leqslant \frac{\mathrm{const}_{p,r}}{m^{r+1}} (\| \mathscr{D}^r f \|_p.$$

Lemma 1 and (3.6) also imply $\| R_{n,r}^2 f - f \|_p \to 0$ as $n \to \infty$, we have

$$\| R_{n,r}^2 f - f \|_p \leqslant \sum_{m=n}^{\infty} \| R_{m,r}^2 f - R_{m+1,r}^2 f \|_p.$$

We finally get Jackson's estimate for $f \in W_p^r$

$$\begin{aligned}
\| R_{n,r} f - f \|_p &\leqslant \| R_{n,r}^2 f - R_{n,r} f \|_p + \sum_{m=n}^{\infty} \| R_{m,r}^2 f - R_{m+1,r}^2 f \|_p \\
&\leqslant \mathrm{const}_{p,r} \left(\frac{1}{(n(n+1))^{r/2}} + \sum_{m=n}^{\infty} \frac{1}{m^{r+1}} \right) \| \mathscr{D}^r f \|_p \\
&\leqslant \frac{\mathrm{const}_{p,r}}{n^r} \| \mathscr{D}^r f \|_p.
\end{aligned}$$

Lemma 3 is proved.

Proof of Theorem 1. Let $f \in L_p[-1, 1]$. Choose $g \in W_p^r$ such that

$$\|f-g\|_p + n^{-r} \|\mathscr{D}^r g\|_p \leqslant 2K(f, n^{-r}; L_p, W_p^r).$$

We get

$$\|R_{n,r}f - f\|_p \leqslant \|R_{n,r}(f-g) - (f-g)\|_p + \|R_{n,r}g - g\|_p$$
$$\leqslant \text{const}_{p,r}\ K(f, n^{-r}; L_p, W_p^r) + \|R_{n,r}g - g\|_p.$$

By making use of Lemma 3, we have

$$\|R_{n,r}g - g\|_p \leqslant \frac{\text{const}_{p,r}}{n^r} \|\mathscr{D}^r g\|_p \leqslant \text{const}_{p,r} K(f, n^{-r}; L_p, W_p^r).$$

Combining the inequalities above we get

$$\|R_{n,r}f - f\|_p \leqslant \text{const}_{p,r} K(f, n^{-r}; L_p, W_p^r).$$

To prove the converse result, by making use of Lemmas 2 and 3 we have

$$\|\mathscr{D}^r R_{n,r} f\|_p \leqslant \text{const}_{p,r}\ n^r \|R_{n,r}f - f\|_p, \qquad f \in L_p[-1, 1].$$

It follows from the definition of K-functional that

$$K(f, n^{-r}; L_p, W_p^r) \leqslant \|f - R_{n,r}f\|_p + n^{-r} \|\mathscr{D}^r R_{n,r} f\|_p$$
$$\leqslant \text{const}_{p,r} \|R_{n,r}f - f\|_p.$$

The proof of Theorem 1 is complete.

From Lemma 2 and the proof of Theorem 1 we deduce that

$$\|R_{n,r}f - f\|_p + n^{-r} \|\mathscr{D}^r R_{n,r} f\|_p \simeq K(f, n^{-r}; L_p, W_p^r).$$

This equivalence relationship shows that the $R_{n,r}f$ can serve as a realization of the K-functional $K(f, n^{-r}; L_p, W_p^r)$.

We now present the relationships between the best polynomial approximant and the generalized Riesz summability operators.

THEOREM 2. *Let* $f \in L_p[-1, 1]$, $1 \leqslant p \leqslant \infty$, *and let* $R_{n,r}f$ *be defined by* (1.8). *Then*

$$E_n(f)_p \leqslant \|R_{n,r}f - f\|_p.$$

conversely,

$$\|R_{n,r}f-f\|_p \leqslant \frac{\text{const}_{p,r}}{n^r} \sum_{0\leqslant k\leqslant n} (k+1)^{r-1} E_k(f)_p.$$

Proof. The first inequality is obvious. Concerning the second one, we have to show the following Bernstein type inequality

$$\|\mathscr{D}^r Q_n\|_p \leqslant \text{const}_{p,r}\ n^r \|Q_n\|_p,$$

where Q_n is a polynomial of order n.

In fact, if Q_n is a polynomial of order n, we can write Q_n as

$$Q_n(x) = \sum_{k=0}^{n} Q_n^{\wedge}(k)\, P_k(x).$$

By the definition of $\mathscr{D}^r$, we get $\mathscr{D}^r Q_n(x) = -\sum_{k=0}^{n} (k(k+1))^{r/2}\, Q_n^{\wedge}(k)\, P_k(x)$. Then the Bernstein type inequality is of the form

$$\left\| \sum_{k=0}^{n} (k(k+1))^{r/2}\, Q_n^{\wedge}(k)\, P_k \right\|_p \leqslant \text{const}_{p,r}(n(n-1))^{r/2} \left\| \sum_{k=0}^{n} Q_n^{\wedge}(k)\, P_k \right\|_p.$$

This is Corollary 5.15 of [8]. The proof of Theorem 2 is complete.

By this theorem and Theorem 1 we have

THEOREM 3. *Let* $f\in L_p[-1,1]$, $1\leqslant p\leqslant\infty$, *and let* $R_{n,r}f$ *be defined by* (1.8). *Then*

$$E_n(f)_p \leqslant \text{const}_{p,r}\, K(f, n^{-r}; L_p, W_p^r).$$

conversely,

$$K(f, n^{-r}; L_p, W_p^r) \leqslant \frac{\text{const}_{p,r}}{n^r} \sum_{0\leqslant k\leqslant n} (k+1)^{r-1} E_k(f)_p.$$

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REFERENCES

1. W. Chen, Z. Ditzian, and K. Ivanov, Strong converse inequality for the Bernstein–Durrmeyer operator, *J. Approx. Theory* **75** (1993), 25–43.
2. Z. Ditzian, A K-functional and the rate of convergence of some linear polynomial operators, *Proc. Amer. Math. Soc.* **124** (1996), 1773–1781.
3. Z. Ditzian and K. Ivanov, Strong converse inequalities, *J. Analyse Math.* **61** (1993), 61–111.
4. Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, Berlin/New York, 1987.
5. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, 1959.
6. V. Totik, Approximation by algebraic polynomials, *in* "Approximation Theory, VII" (E. W. Cheney, C. K. Chui, and L. L. Schumaker, Eds.), pp. 227–249, Academic Press, Boston/San Diego/New York, 1993.
7. V. Totik, Approximation by Bernstein polynomials, *Amer. J. Math.* **116** (1994), 995–1018.
8. W. Trebels, Multipliers for (C, α)-bounded Fourier expansions in Banach spaces and approximation theory, *in* "Lecture Notes in Math.," Vol. 329, Springer-Verlag, Berlin/New York, 1973.